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# Universal conductance distribution in three-dimensional systems in high magnetic fields

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**Abstract.** The nature of the critical point of the Anderson transition in high magnetic fields is discussed with an emphasis on scale invariance and universality of the critical exponent. Special attention is paid to the distribution function of the conductance which becomes size and model independent at the critical point. The fractal properties of the wave function which are related to scale invariance are also discussed.

## 1. Introduction

The Anderson transition (AT) has been attracting the attention of condensed matter physicists for more than four decades [1, 2]. The AT is a zero-temperature quantum phase transition separating metallic and insulating phases which is induced by a spatially fluctuating random potential. The transition can be described using the scaling theory of localization [3, 4]. Near the critical point in three-dimensional (3D) systems, behaviour which is typical of quantum phase transitions is observed for quantities such as the conductance $g$ and correlation length $\xi$. For example, as we approach the critical point by changing a parameter $w$, such as the strength of disorder or the Fermi energy, the correlation length diverges as

$$\xi \sim |w - w_c|^{-\nu} \tag{1}$$

while the conductivity $\sigma$ vanishes from the metallic side according to the power law

$$\sigma \sim |w - w_c|^{s}. \tag{2}$$

If we approach the transition from the insulating side, then the dielectric constant $\epsilon$ diverges as

$$\epsilon \sim |w - w_c|^{-s'}. \tag{3}$$

As in the critical phenomena of magnetic systems, the exponents $\nu$, $s$ and $s'$ are not independent but are related [5]:

$$s = (d-2)\nu \qquad s' = 2\nu. \tag{4}$$

A knowledge of $\nu$ is enough to fix the critical exponents $s$ and $s'$. These latter exponents can be measured experimentally though there has been controversy concerning the correct values [6]. It is thus important to have a precise theoretical estimate of $\nu$ in order to compare with the experiments.

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The value of $\nu$ is expected to be universal, i.e., independent of the details of the model and dependent only on basic symmetries such as that under the operation of time reversal. The classification of the critical behaviour according to the symmetries of the system was predicted from field-theoretic considerations [7, 8], and recently verified numerically [9].

At the Anderson transition the correlation length diverges and the wave function becomes scale invariant. This invariance is characterized by a fractal dimension $D_2$. This is reflected in the size independence of the distribution function of the conductance [9, 10] as well as the statistics of the energy levels [11–15].

In this paper we report a numerical simulation of the wave-function dynamics at the critical point for the tight-binding model in a magnetic field. We have observed anomalous diffusion and estimated the fractal dimensionality. The universality of the distribution function of the conductance at the critical point has also been verified. The final section is devoted to a summary of our results and concluding remarks in connection with experiments.

## 2. Anderson transition in magnetic fields

Magnetic fields have two effects on the Anderson transition. One is to delocalize the electronic states by breaking time-reversal symmetry, and the other is to localize the electronic states by shrinking the wave function due to cyclotron motion. Which of these two dominates depends on the situation.

The tight-binding Hamiltonian which incorporates the effect of magnetic fields is given by

$$H = V \sum_{\langle i,j \rangle} \exp(\mathrm{i}\theta_{i,j}) C_i^\dagger C_j + \sum_i W_i C_i^\dagger C_i \tag{5}$$

where $C_i^\dagger$ ($C_i$) denotes a creation (annihilation) operator of an electron at the site $i$. The energies $\{W_i\}$ are distributed independently and uniformly in the range $[-W/2, W/2]$. By fixing the Fermi energy $E$ to be, e.g., $E = 0$ while increasing $W$, the system is driven to be an insulator at $W = W_c$ where $W_c$ is the critical disorder. The Peierls phase factors $\exp(\mathrm{i}\theta_{i,j})$ describe magnetic fields. The hopping amplitude $V$ is assumed to be the energy unit, $V = 1$. We assume a simple cubic 3D lattice for simplicity, and all of the length scales are measured in units of its lattice constant $a$.

In the absence of magnetic fields, the AT occurs at a critical disorder $W_c \simeq 16.5$ at the centre of the band, $E = 0$. The critical exponent $\nu$ has been estimated as $\nu = 1.59 \pm 0.03$ [9]. Applying strong magnetic fields to the tight-binding model, it has been shown that the value of the critical point as well as the scaling curve change. In a magnetic field the value of $\nu$ is estimated to be $1.43 \pm 0.04$ [9]. The exponent is not dependent on the strength of the magnetic fields [9] and is unchanged in random magnetic fields which can be realized by assuming a random phase for the hopping elements [16].

Another model which describes the 3D Anderson transition in a high magnetic field is a stack of two-dimensional layers with a strong quantizing field applied perpendicular to the plane [17]. In purely two-dimensional systems, the quantum Hall effect (QHE) occurs. The electronic states are delocalized only at the centre of each Landau band. In this case the critical exponent $\nu_{\mathrm{QHE}}$ is estimated to be close to 7/3 [18]. The introduction of interlayer hopping between the layers makes the delocalized region finite in energy, and changes the exponent to $1.45 \pm 0.15$ [19]. This is close to the value of 1.43 given above, in agreement with universality of the AT. Universality has also been verified for changes in the transfer integral between the layers [17].

There is another interesting aspect of the stacked-layer model. If we impose Dirichlet boundary conditions instead of periodic boundary conditions, we find magnetic edge states circulating along the perimeter of the two-dimensional layer. Stacking the layers along the $z$-direction, we then have novel electronic states where the electron can hop both in the $+z$- and $-z$-directions but can rotate only in the clockwise or anticlockwise direction (which is determined by the direction of the field) in the $x$–$y$ plane. The edge states which compose this strange 'sheath' are critical [19, 20]. That is to say, as we increase the linear dimension of the plane, say $L$, the localization length in the $z$-direction $\xi_{\|}$ also increases in proportion to $L$ and diverges in the thermodynamic limit.

## 3. Critical behaviour

### *3.1. Anomalous diffusion and fractal dimensionality*

In the metallic regime, the electron diffuses and the mean squared diffusion radius

$$\overline{r^2(t)} \equiv \overline{\langle t|\boldsymbol{r}^2|t\rangle}$$

is proportional to the time $t$:

$$\overline{r^2(t)} = 2dDt \tag{6}$$

where $D$ is the diffusion coefficient and the average is taken over disorder. In the insulating regime where the wave function $\psi$ is localized as $\exp(-r/\xi)$, the squared diffusion radius saturates at [21]

$$\lim_{t\to\infty} \overline{r^2(t)} = \frac{d(d+1)}{4}\xi^2. \tag{7}$$

To understand the intermediate region we use the renormalization group, from which

$$\overline{r^2(t)} = b^2 f((w-w_c)b^{1/\nu}, tb^{-z})$$

where $b$ is the scale factor in the renormalization group and $z$ is the dynamical exponent. The time $t$ is measured in units of $\hbar/V$. From this equation we deduce the scaling form [22]

$$\overline{r^2(t)} = t^{2/z} F(t^{1/z\nu}(w-w_c)). \tag{8}$$

A similar relation holds for classical percolation theory [23]. On the metallic side of the transition we expect at sufficiently long times a linear-in-$t$ growth of the mean squared radius in accordance with (6). For this to occur, we must have $F(x) \sim x^s$ when $x \gg 1$ with $s = (z-2)\nu$. Thus on the metallic side of the transition at long times, we have

$$\overline{r^2(t)} \sim |w-w_c|^s t.$$

Since according to the Einstein relation $\sigma \sim D$, we see that $s$ is indeed the exponent in (2). For non-interacting electrons $z = d$, and we recover the Wegner scaling law [5]

$$s = (d-2)\nu.$$

On the insulating side of the transition we expect (7) to hold at long times. Imposing this in (8) leads to

$$\overline{r^2(t)} \sim |w-w_c|^{-2\nu} \sim \xi^2$$

confirming that $\nu$ in (8) is indeed the exponent governing the divergence of the localization length. Exactly at the critical point, $w = w_c$, and we see that the square diffusion length grows as

$$\overline{r^2(t)} \sim t^{2/3}. \tag{9}$$

Another quantity which can be used to investigate the dynamics of the wave function at the critical point is the return probability

$$C(t) \equiv \frac{1}{t}\int_0^t \mathrm{d}t' \ |\langle t'|0\rangle|^2 \tag{10}$$

which is related to the fractal dimensionality of the wave function $D_2$ as

$$C(t) \sim t^{-D_2/d}. \tag{11}$$

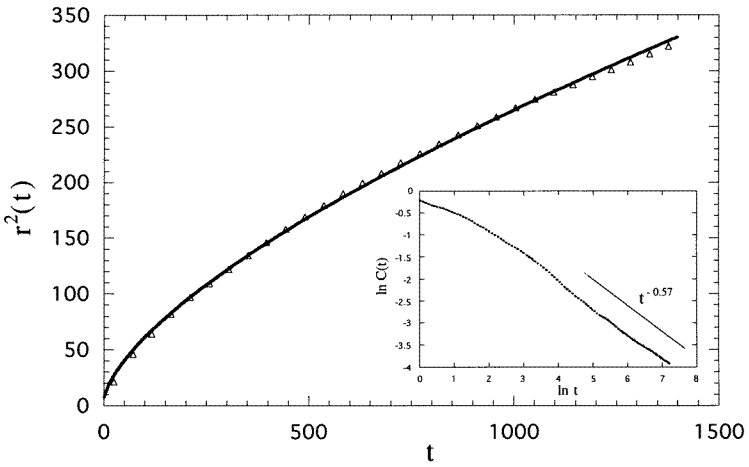


**Figure 1.** The squared diffusion length $r^2(t)$ versus time $t$. The solid line is the fit to $t^{2/3}$. Inset: a double-logarithmic plot of the return probability $C(t)$ versus $t$. The line representing the power law $t^{-0.57}$ is a guide to the eyes.

Direct diagonalization of the 3D systems requires huge CPU power, especially when the Hamiltonian is complex, because of the applied magnetic fields. Instead of carrying out direct diagonalization, we have used the equation-of-motion method to study the diffusion process. We prepare a wave packet $|0\rangle$ with the energy $E$ located at the centre of the system, and numerically calculate the time evolution using $|t\rangle = \mathrm{e}^{-\mathrm{i}Ht/\hbar}|0\rangle$. When evaluating the factor $\mathrm{e}^{-\mathrm{i}Ht/\hbar}$, we use the decomposition formula for exponential operators [24, 25]. In figure 1, we show the results for $\bar{r^2}(t)$ calculated for a $59 \times 59 \times 59$ cubic lattice. The magnetic field is parallel to the $z$-direction and the magnitude of the flux per unit cell is 0.1 times the flux quantum. The critical disorder in this case is $W_c = 17.8$ [26]. We see clearly the $t^{2/3}$-law for $r^2(t)$, confirming the validity of the scaling equation (8) as well as the scaling relation given by equation (4). The estimate of $D_2$ from $C(t)$ is shown in the inset of figure 1. We find $D_2 = 1.7$ which is significantly smaller than the spatial dimension 3, demonstrating that the wave function at the transition is not at all similar to a typical extended wave function. This value is consistent with the recent estimate of $D_2$ for layered systems in high magnetic fields [27].

### *3.2. Conductance distribution*

In a $d$-dimensional hypercubic lattice, the dimensionless conductance $g$ is defined as

$$g = \frac{G}{e^2/h} = \frac{\sigma L^{d-2}}{e^2/h}. \tag{12}$$

From the Landauer formula [28], we have

$$g = \mathrm{tr}\,\mathbf{t}\mathbf{t}^{\dagger} \tag{13}$$

where $\mathbf{t}$ is the transmission matrix. The matrix $\mathbf{t}$ is obtained by iteratively calculating the Green function [29].

In 3D metallic $L \times L \times L$ systems, the conductance distribution function $P(g)$ is the normal distribution, the mean of which, $\langle g \rangle$, is proportional to the size $L$. The variance, on the other hand, is universal, a phenomenon which is known as universal conductance fluctuations [30]. In the insulating regime, $P(g)$ is log–normal.

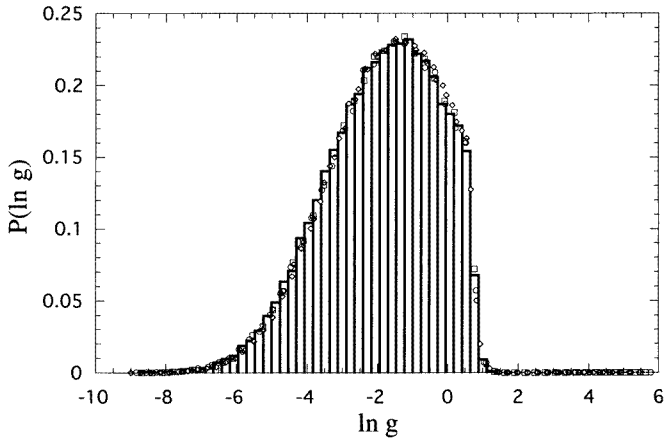


**Figure 2.** The distribution function of the logarithm of the conductance $g$ at the critical point. Triangles (△), diamonds (◇), squares (□) and circles (○) correspond to $L = 8$, 10, 12 and 14 for the random-phase hopping model, respectively. The histogram is for a system in uniform magnetic fields.

At the Anderson transition, not only the variance but also the distribution function itself becomes universal [9, 10, 31]. The resulting distribution function is plotted in figure 2 for $L = 8$, 10, 12 and 14. We plot $P(\log g)$ instead of $P(g)$ to show more clearly the detail of the distribution function. The histogram is for the uniform magnetic field while the dots are for the random-phase hopping model. The critical disorder $W_c$ depends on the strength of the field and how we break the time-reversal symmetry (i.e., with a uniform magnetic field or random-phase hopping). Nevertheless, $P(\log g)$ at the critical points is universal.

Once the system is away from the critical point, the $P(g)$ begins to show size dependence. To demonstrate this size dependence, we plot $P(\log g)$ in vanishing field away from criticality at $W = 17.5$ in figure 3. This value of the disorder is about 6% larger than $W_c(B = 0) = 16.5$. We can see subtle but clear size dependence of $P(g)$.

Similar behaviour of the conductance distribution is also observed in the layered system in high magnetic fields, though in this model the system is highly anisotropic and the form of $P_c(g)$ is different [32].

## 4. Summary and concluding remarks

In this paper we have discussed several features of the Anderson transition which are related to the self-similarity of the eigenstates at the critical point. The fractal dimensionality of these critical eigenstates is almost half the original space dimension, $\approx$1.7. The square

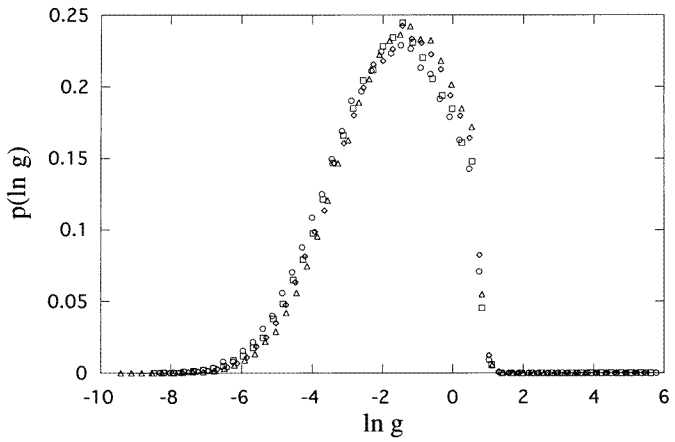


**Figure 3.** The distribution function of the logarithm of the conductance $g$ when the strength of the disorder is slightly larger than the critical value $W_c$. Triangles (△), diamonds (◇), squares (□) and circles (○) again correspond to $L = 8$, 10, 12 and 14, respectively.

diffusion length $r^2(t)$ has been shown to grow as $t^{2/3}$ in the non-interacting model, irrespective of the values of the critical exponents. At the transition, the distribution function of the conductance $P(g)$ becomes model and size independent. The distribution function of $g$ close to AT had been recently obtained experimentally [33], and is consistent with our results. Such universality of the distribution is also seen in the statistics of the energy levels at the transition [11–15].

We have also seen that the layered system in perpendicular magnetic fields shows interesting transport properties. The critical behaviour of this system may still be the same as that found in the tight-binding isotropic system.

In order to relate our results to experiments at finite temperature $T$, we now discuss the $T$-dependence of the conductivity, $\sigma(T)$. At finite temperature, the inelastic scattering time $\tau_{\rm in}$ and the inelastic scattering length $l_{\rm in}$ are finite. In the metallic regime, they are related by $l_{\rm in} \sim \tau_{\rm in}^{1/2}$. At criticality, the diffusion is anomalous, and this becomes

$$l_{\rm in} \sim \tau_{\rm in}^{\nu/(s+2\nu)}. \tag{14}$$

The effective diffusion coefficient $D_{\rm eff}$ becomes

$$D_{\rm eff} \sim \frac{l_{\rm in}^2}{\tau_{\rm in}} \sim \tau_{\rm in}^{-s/(s+2\nu)} \tag{15}$$

leading to the conductivity at finite temperature $\sigma(T)$:

$$\sigma(T) \sim T^{s/(s+2\nu)} \tag{16}$$

where we have assumed that $\tau_{\rm in} \sim 1/T$. Setting $s = \nu$ gives $\sigma \sim T^{1/3}$ which is independent of the values of the exponents $s$ and $\nu$. In the presence of electron–electron interaction, relation (4) may no longer be valid and instead

$$s = (d - 2 - \theta)\nu \tag{17}$$

should be used [34]. This leads to the suggestion that the exponent of the temperature dependence is different when the time-reversal symmetry is broken. In experiments [35], the power $1/3$ is widely observed. This means that even in the presence of electron–electron interaction, the relation $s = \nu$ may not be modified significantly.

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